Green's function of wave field in media with one-dimensional large-scale periodicity

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The Green's function of the wave field in a medium with a smooth one-dimensional periodicity is considered. The solution is constructed by the WKB method. It is shown that at large distances there is an analogy between the Green's function in a medium with one-dimensional periodicity and the Green's function in an anisotropic uniaxial medium. The periodic system is distinguished from an anisotropic medium by a discontinuity of the wave vector surface and a break of beam vector surface. The forbidden zone corresponds to capture of beams with small angles of incidence and formation of a wave guide channel. Within this wave guide channel the Green's function asymptotic differs from $1/r$ behavior. The fields outside and inside of the wave channel are described within the framework of a unique approach. A detailed analysis of the obtained results is carried out. [S1063-651X(98)15912-3]

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I. INTRODUCTION

There exist a set of physical systems with properties periodically varying in space. Studying the propagation and scattering of waves is one of the most effective methods of such systems investigations.

Rather frequently we deal with systems possessing one-dimensional periodicity. In particular, these problems occur in acousto-optics [1], holography [2], integrated and optical electronics [3], x-ray diffractive optics [4], and propagation of waves in multilayer coverings.

Recently the special interest appears to be research into electromagnetic wave propagation in three-dimensional periodic dielectric structures (photonic band structures) [5–8]. Much attention is devoted to research into anisotropic and hydrotropic periodic media. In particular, many interesting results are obtained for the optics of liquid crystals [9].

When studying the propagation of waves in periodic systems the basic attention is paid to eigenwaves. However, the solution of many problems requires knowledge of the Green's function of the wave equation, i.e., the field of a point source. Particular attention has been paid to studying Green's functions for systems with a complex structure [5,10–14].

The medium with one-dimensional periodicity was considered in [10,11,13]. The complexity of the problem can be seen in the fact that, in contrast to cases of the ordinary isotropic and anisotropic media, in our case the Green's function depends on both vector arguments separately. For a weakly inhomogeneous medium with a small-scale periodicity the Green's function was considered in [11].

In the present work we study systems with an arbitrary amplitude of periodicity for cases of period being large compared to the wavelength. The Green's function is calculated for any distances from the source. The case of large distances is analyzed in detail. It is shown that forbidden zones exist in such media. For directions outside of the forbidden zones the structure of the field is similar to one in the media with uniaxial anisotropy. In particular, a surface of wave vectors is not spherical and the directions of wave and beam vectors do not coincide. Particular attention is paid to the study of the forbidden zones and propagation of waves in a wave channel. The results are illustrated by numerical calculations.

The paper is organized as follows. General equations and various approaches for solution of considered problem are presented in Sec. II. In Sec. III we consider the Green's function in periodic media with a large-scale periodicity. In Sec. IV the origin of forbidden zones is discussed. In Sec. V we study the wave propagation in the wave channel.

II. GENERAL EQUATIONS

Let us consider the propagation of waves in an isotropic inhomogeneous medium. In what follows we are not interested in polarization effects. Then the field of the wave $u(\mathbf{r},t)$ obeys the scalar equation

$$\left(\Delta-\frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\right)u(\mathbf{r},t)=0, \tag{1}$$

where $c(\mathbf{r})$ is the velocity of the wave and $u(\mathbf{r})$ is the amplitude of the wave.

For harmonic time dependence, $u(\mathbf{r},t)=u(\mathbf{r})e^{-i\omega t}$, we arrive at the Helmholtz equation

$$[\Delta+k^2(\mathbf{r})]u(\mathbf{r})=0, \tag{2}$$

where $k(\mathbf{r})=\omega/c(\mathbf{r})$ is the wave number.

Let the medium be periodic along the z direction with the period d. Then $k^2(\mathbf{r})$ can be presented in the form

$$k^2(\mathbf{r})=k_0^2[1+f(z)], \tag{3}$$

where $f(z)$ is a periodic function, $f(z+d)=f(z)$.

The Green's function satisfies the following equation:

$$[\Delta+k_0^2(1+f(z))]T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\delta(\mathbf{r}-\mathbf{r}_1) \tag{4}$$

and the condition of radiation. Here $\boldsymbol{\rho}=(x,y)$ is the component of the $\mathbf{r}$ vector normal to the z axis, and $\delta(\mathbf{r})$ is a three-dimensional δ function. Due to spatial periodicity along the z axis the Green's function depends on both arguments z and z_1 separately.

Since the medium is uniform in the transverse plane xy we use a two-dimensional Fourier-transform of Eq. (4),

$$\left[\frac{\partial^2}{\partial z^2}-q^2+k_0^2(1+f(z))\right]T(\mathbf{q};z,z_1)=\delta(z-z_1). \qquad (5)$$

Here

$$T(\mathbf{q};z,z_1)=\int d\boldsymbol{\rho}T(\boldsymbol{\rho};z,z_1)\exp(-i\mathbf{q}\boldsymbol{\rho}). \qquad (6)$$

Equation (5) with the radiation condition is the Sturm-Liouville problem. Therefore it is possible to use the well-known theorems. Namely, the Green's function can be written as

$$T(\mathbf{q};z,z_1)=\frac{1}{W}\times\begin{cases}u_1(z)u_2(z_1) & \text{for } z\geqslant z_1,\\ u_1(z_1)u_2(z) & \text{for } z<z_1,\end{cases} \qquad (7)$$

where $u_1(z),u_2(z)$ are independent solutions of the corresponding homogeneous equation,

$$\left[\frac{\partial^2}{\partial z^2}-q^2+k_0^2(1+f(z))\right]u(z)=0, \qquad (8)$$

with $W=u_1(z)u_2'(z)-u_1'(z)u_2(z)$ being the Wronskian of functions $u_1(z)$ and $u_2(z)$. Value W does not depend on z since Eq. (5) does not contain the first z derivative.

In order to construct the Green's function satisfying the radiation condition it is necessary to take $u_1(z)$ describing the wave propagating at $z\rightarrow+\infty$, and $u_2(z)$ should describe similarly the wave propagating along the $z\rightarrow-\infty$ direction. Equation (8) is the Hill equation and, according to the Floquet theorem the solutions $u_1(z)$ and $u_2(z)$ can be written as [15]

$$u_1(z)=\Phi_1(z)e^{i\mu z}, \quad u_2(z)=\Phi_2(z)e^{-i\mu z}, \qquad (9)$$

where $\Phi_1(z)$ and $\Phi_2(z)$ are periodic functions of z with the period d, and μ is a constant. The functions $\Phi_i(z)$, $(i=1,2)$ and constant μ are determined by the q parameter. For certain values of q the constant μ becomes complex and waves $u_1(z)$ and $u_2(z)$ decay and increase, respectively. The region of these values of q corresponds to the so-called forbidden zones [16]. In coordinate representation these regions yield exponentially small contributions to the Green's function asymptotics at $|\mathbf{r}-\mathbf{r}_1|\rightarrow\infty$. An exception is the case when $\mathbf{r}$ and $\mathbf{r}_1$ are in the same layer.

We consider an inhomogeneous medium with $f(z)=\eta\cos(2\alpha z)$, where $\alpha=\pi/d$, $\eta<1$ is a factor, describing an amplitude of nonhomogeneity, and d is the period of the structure. In this case Eq. (8) is reduced to the Mathieu equation

$$\left[\frac{\partial^2}{\partial z^2}-q^2+k_0^2+k_0^2\eta\cos(2\alpha z)\right]u(z)=0. \qquad (10)$$

The formal solution of the Hill equation is known [16], but obtaining relevant numerical results is a difficult problem. Even in the case of the Mathieu equation consistent analysis requires introducing of small parameters. Namely, in Ref. [11] a case of weakly nonhomogeneous media with $\eta\ll 1$ were considered. Due to the evenness of the $f(z)$ function it is possible to take $u_1(-z)$ for the $u_2(z)$ function. In this case the solution (7) has the form

$$T(\mathbf{q};z,z_1)=\frac{1}{W}e^{i\mu|z-z_1|}\times\begin{cases}\Phi(z)\Phi(-z_1) & \text{at } z\geqslant z_1,\\ \Phi(z_1)\Phi(-z) & \text{at } z<z_1.\end{cases} \qquad (11)$$

The parameter μ and function $\Phi(z)$ were sought within the framework of the perturbation theory for systems with the period of structure of the order of a wavelength, i.e., for $k_0\sim\alpha$. In such systems forbidden zones exist in the case $k_0>\alpha$. The account of fields in the vicinity of forbidden zones requires modification of the perturbation theory. When parameter k_0/α is large the number of forbidden zones increases infinitely whereas their width correspondingly tends to zero. In this case applying the Floquet theorem to description of the field becomes inconvenient. On the other hand, at smooth periodicity the parameters of the medium vary insignificantly at a distance of a wavelength and the WKB method becomes more suitable for description of the field.

III. GREEN'S FUNCTION IN A MEDIUM WITH A LARGE-SCALE PERIODICITY

We consider a case when parameter η is not small, and the period of structure d is much greater than the wavelength $\lambda=2\pi/k_0$, that is, $\Omega=k_0/\alpha\gg 1$.

Let us introduce the dimensionless variable $\xi=\alpha z$. Then Eq. (10) is written as

$$\left[\frac{\partial^2}{\partial\xi^2}-\frac{q^2}{\alpha^2}+\Omega^2(1+\eta\cos 2\xi)\right]u(\xi)=0. \qquad (12)$$

Solving Eq. (12) by the WKB method we get in the first approximation

$$u_{1,2}(z)=\Gamma^{-(1/2)}(q,\alpha z)\exp\left(\pm i\Omega\int_{\alpha z_0}^{\alpha z}d\xi'\Gamma(q,\xi')\right), \qquad (13)$$

where z_0 is an arbitrary coordinate determining amplitudes of the functions $u_1(z)$ and $u_2(z)$ and function $\Gamma(q,\xi)$ is equal to

$$\Gamma(q,\xi)=\sqrt{1-\frac{q^2}{k_0^2}+\eta\cos 2\xi}. \tag{14}$$

It is easy to verify that the Wronskian of the functions $u_1(z)$ and $u_2(z)$ does not depend on z and is determined as $W=-2ik_0$. Then Green's function (7) has the form

$$T(\mathbf{q};z,z_1)=\frac{i}{2k_0\Gamma^{1/2}(q,\alpha z)\Gamma^{1/2}(q,\alpha z_1)}\times\exp\left(i\Omega\left|\int_{\alpha z_1}^{\alpha z}d\xi\Gamma(q,\xi)\right|\right). \tag{15}$$

In coordinate representation we get

$$T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\frac{i}{8\pi^2k_0}\int d\mathbf{q}\times\frac{\exp[i(\Omega|\int_{\alpha z_1}^{\alpha z}d\xi\Gamma(q,\xi)|+\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1))]}{\Gamma^{1/2}(q,\alpha z)\Gamma^{1/2}(q,\alpha z_1)}. \tag{16}$$

Equation (16) describes the Green's function at any distances. We are interested in the asymptotics of the Green's function in the far zone. So we calculate the integral in Eq. (16) for $|\mathbf{r}-\mathbf{r}_1|\gg\lambda$ by the method of the stationary phase [17]. In order to do this it is necessary to find a stationary point $\mathbf{q}_{st}$ and to expand the exponent in Taylor series over $\mathbf{p}=(\mathbf{q}-\mathbf{q}_{st})$ up to terms of second order. In all nonexponential factors it is possible to set $\mathbf{q}=\mathbf{q}_{st}$. The remaining Gaussian integral is easily calculated:

$$\int\exp\left(\frac{i}{2}\mathbf{p}\hat{H}\mathbf{p}\right)d\mathbf{p}=\frac{2\pi i\sigma}{\sqrt{|\det\hat{H}|}}. \tag{17}$$

Here $\hat{H}$ is the real symmetric matrix of the second derivatives calculated at $\mathbf{q}=\mathbf{q}_{st}$ (Hesse matrix). The factor σ in Eq. (17) depends on signs of real eigenvalues μ_1 and μ_2 of the $\hat{H}$ matrix: for $\mu_1>0$, $\mu_2>0$, $\sigma=1$; for $\mu_1\mu_2<0$, $\sigma=-i$; for $\mu_1<0$, $\mu_2<0$, $\sigma=-1$ [17].

The stationary point $\mathbf{q}_{st}$ is determined from the equation

$$\nabla_q\left(\Omega\left|\int_{\alpha z_1}^{\alpha z}d\xi\Gamma(q,\xi)\right|+\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)\right)=0. \tag{18}$$

Equation (18) may be rewritten as

$$\frac{k_0^2|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}=\Omega\left|\int_{\alpha z_1}^{\alpha z}\frac{d\xi}{\Gamma(q_{st},\xi)}\right|,\quad \mathbf{q}_{st}=q_{st}\frac{\boldsymbol{\rho}-\boldsymbol{\rho}_1}{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}. \tag{19}$$

We take into account that the primitive of a periodic function can be presented as a sum of a linear function and a periodic one with the same period, as that for the initial function [15]. In what follows we will use the Legendre elliptic integrals [18]

$$K(a)=\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-a^2\sin^2\psi}},$$

$$E(a)=\int_0^{\pi/2}d\psi\sqrt{1-a^2\sin^2\psi}, \tag{20}$$

$$F(\phi,a)=\int_0^{\phi}\frac{d\psi}{\sqrt{1-a^2\sin^2\psi}}.$$

Then the solution of Eq. (19) may be written as

$$\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}=\frac{2|z-z_1|}{\pi\sqrt{k_0^2-q_{st}^2+\eta k_0^2}}\times K\left(\sqrt{\frac{2\eta k_0^2}{k_0^2-q_{st}^2+\eta k_0^2}}\right)+B(z,z_1), \tag{21}$$

where $B(z,z_1)$ is the periodic function,

$$B(z,z_1)=\frac{1}{\alpha k_0\Gamma(q_{st},0)}\times\left|F\left(\alpha z,\frac{2\eta}{\Gamma(q_{st},0)}\right)-F\left(\alpha z_1,\frac{2\eta}{\Gamma(q_{st},0)}\right)\right|-\frac{2|z-z_1|}{\pi k_0\Gamma(q_{st},0)}K\left(\frac{2\eta}{\Gamma(q_{st},0)}\right). \tag{22}$$

As seen from Eq. (22) the stationary point q_{st} depends on the difference vector $\mathbf{r}-\mathbf{r}_1$ and coordinates z and z_1 entering into function $B(z,z_1)$ separately.

The Hesse matrix in Eq. (17) has the form

$$H_{\alpha\beta}(q)=\nabla_{q_\beta}\nabla_{q_\alpha}\left(\Omega\left|\int_{\alpha z_1}^{\alpha z}d\xi\Gamma(q,\xi)\right|+\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)\right)=-\frac{\Omega}{k_0^2}\left|\int_{\alpha z_1}^{\alpha z}\frac{d\xi}{\Gamma(q,\xi)}\left(\delta_{\alpha\beta}+\frac{q_\alpha q_\beta}{k_0^2\Gamma^2(q,\xi)}\right)\right|. \tag{23}$$

The Hessian is equal to

$$\det\hat{H}=\frac{\Omega^2}{k_0^4}\left|\int_{\alpha z_1}^{\alpha z}\frac{d\xi}{\Gamma(q_{st},\xi)}\right|\times\left|\int_{\alpha z_1}^{\alpha z}\frac{d\xi}{\Gamma(q_{st},\xi)}\left(1+\frac{q_{st}^2}{k_0^2\Gamma^2(q_{st},\xi)}\right)\right|. \tag{24}$$

According to Eq. (23), in this case $\mu_1<0$ and $\mu_2<0$ and factor $\sigma=-1$ in Eq. (17). Using Eqs. (16), (17), and (24) we arrive at the expression for the Green's function in the far zone:

$$T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)$$
$$=\frac{\exp(i\Omega|\int_{\alpha z_1}^{\alpha z}d\xi\Gamma(q_{st},\xi)|+iq_{st}|\boldsymbol{\rho}-\boldsymbol{\rho}_1|)}{4\pi k_0\Gamma^{1/2}(q_{st},\alpha z)\Gamma^{1/2}(q_{st},\alpha z_1)}$$
$$\times\left[\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}\left(\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}+\left|\int_{z_1}^{z}\frac{q_{st}^2dz'}{k_0^3\Gamma^3(q_{st},\alpha z')}\right|\right)\right]^{-(1/2)}. \tag{25}$$

Deriving this expression we have simplified determinant (24) using Eq. (19).

We analyze the behavior of the Green's function at distances $|z-z_1|$ considerably exceeding the period d. In this case function $B(z,z_1)$ in Eq. (21) may be omitted, and the exponent in the expression for the Green's function (25) is

$$i\left[\frac{2}{\pi}|z-z_1|\sqrt{k_0^2-q_{st}^2+\eta k_0^2}\right.$$
$$\left.\times E\left(\sqrt{\frac{2\eta k_0^2}{k_0^2-q_{st}^2+\eta k_0^2}}\right)+q_{st}|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\right]. \tag{26}$$

Equation (21) shows that in this case q_{st} depends on relation $|z-z_1|/|\boldsymbol{\rho}-\boldsymbol{\rho}_1|$ only. Presenting Eq. (26) in the form

$$i[q_z(z-z_1)+\mathbf{q}_{st}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)], \tag{27}$$

where

$$q_z=\frac{2}{\pi}\text{sign}(z-z_1)\sqrt{k_0^2-q_{st}^2+\eta k_0^2}E\left(\sqrt{\frac{2\eta k_0^2}{k_0^2-q_{st}^2+\eta k_0^2}}\right), \tag{28}$$

one can see that three-dimensional vector $\mathbf{k}_{st}=(\mathbf{q}_{st},q_z)$ may be considered as a wave vector. Since this vector does not depend on absolute values of coordinates z and z_1 the properties of the medium become similar to that for the spatially homogeneous media. The Green's function in cases $|z-z_1|\gg d$ is

$$T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\frac{\exp[i\mathbf{k}_{st}(\mathbf{r}-\mathbf{r}_1)]}{4\pi R(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)}, \tag{29}$$

where

$$R(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=k_0\Gamma^{1/2}(q_{st},\alpha z)\Gamma^{1/2}(q_{st},\alpha z_1)$$
$$\times\left[\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}\left(\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{q_{st}}+\frac{2q_{st}^2|z-z_1|}{\pi k_0^3}\right.\right.$$
$$\left.\left.\times\int_0^{\pi/2}\frac{d\xi'}{\Gamma^3(q_{st},\xi')}\right)\right]^{1/2}.$$

For a weakly inhomogeneous medium, i.e., at small value of η, Eq. (19) for q_{st} and the expression for Green's function (25) admit additional simplifications. In lowest order in η Eq. (19) is easily solved and we get

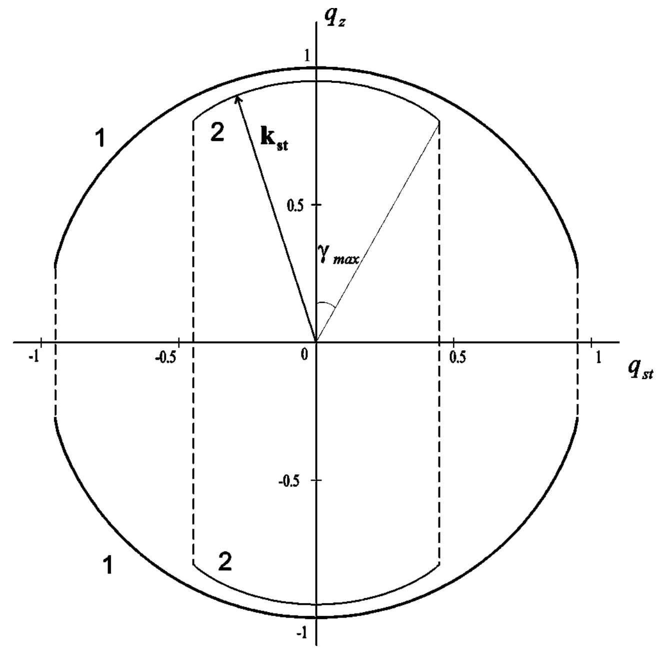

FIG. 1. Cross sections of a surface of wave vectors by a plane, containing the z axis, for different depths of modulation η: 1, $\eta=0.1$; 2, $\eta=0.8$ for $k_0/\alpha=20$. The dotted line shows the forbidden zones; γ_{max} is the maximal possible angle between the wave vector and z axis for $\eta=0.8$. It is seen that the value of k_{st} determining the phase speed of a wave weakly depends on the wave vector direction. All wave numbers are expressed in terms of k_0.

$$q_{st}=\frac{k_0|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{|\mathbf{r}-\mathbf{r}_1|}\left(1+\frac{\eta|\sin 2\alpha z-\sin 2\alpha z_1|}{4\alpha|z-z_1|}\right). \tag{30}$$

In this case the phase of the Green's function has the form

$$k_0|\mathbf{r}-\mathbf{r}_1|\left(1+\frac{\eta|\sin 2\alpha z-\sin 2\alpha z_1|}{4\alpha|z-z_1|}\right). \tag{31}$$

These formulas are valid if the condition $\eta|\mathbf{r}-\mathbf{r}_1|^2\ll 4(z-z_1)^2$ is fulfilled. Note that with $\eta\to 0$ the amplitude factor $R(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)\to|\mathbf{r}-\mathbf{r}_1|$ and the Green's function (29) passes into that for a homogeneous medium. Equation (31) shows that the constant phase regions for the Green's function are the spherical surfaces with small periodic distortions. The corrections in Eqs. (30) and (31) depend not only on the difference $|z-z_1|$, but also on values of z and z_1.

The spatial periodicity at large distances leads to deviation of the wave vector surfaces, Eq. (28), from spherical ones. As long as the medium has an axial symmetry, these surfaces are similar to those with the wave vectors of uniaxial anisotropic media having the form of ellipsoid [19]. However, in our case they are more complicated.

Figure 1 shows the y-z plane cross section of a surface of wave vectors, calculated from Eq. (28) for two values of parameter η that describes the depth of modulation: $\eta=0.1$ and 0.8. One can see that these surfaces are discontinuous and the breaks are increased with growth of η. This break corresponds to the forbidden zone, i.e., to the restriction for possible directions of wave vectors $\mathbf{k}_{st}$. Formally emergence of the forbidden zones is a consequence of the functions $E(a)$ as well as $K(a)$ and $F(\phi,a)$ in Eq. (20)

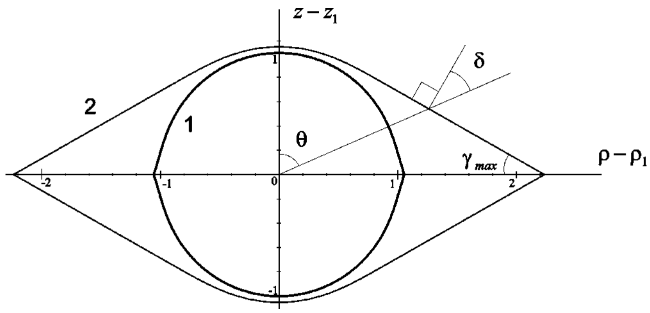

FIG. 2. Constant phase surfaces of waves radiated by a point source. Curves are calculated for the same parameters, as in the previous figure. One can see that at large distances the group speed of a wave along the direction of the periodic structure is appreciably less than in the transverse direction. Here γ_{max} is the limiting angle of the normal inclination to this surface and δ is the angle between wave and beam vectors.

being real only for $|a|<1$. As seen from Eq. (28) this restriction leads to inequalities $q_{st}^2 \leqslant q_{max}^2 = k_0^2(1-\eta)$ and $q_z \geqslant q_{min} = (2/\pi)k_0\sqrt{2\eta}$. If $q_{st} > q_{max}$ the function $E(a)$ and consequently q_z become complex, and the waves decay with the extinction length of the order of the wavelength. Let γ be the angle between $\mathbf{k}_{st}$ and axis z:

$$\tan\,\gamma = \frac{q_{st}}{q_z}. \tag{32}$$

Then the restriction on possible directions of wave vectors $\mathbf{k}_{st}$ can be written as $\pi/2-\gamma_{max} \leqslant |\gamma-\pi/2| \leqslant \pi/2$, where

$$\gamma_{max} = \arctan\frac{q_{max}}{q_{min}} = \arctan\frac{\pi}{2}\sqrt{\frac{1-\eta}{2\eta}}. \tag{33}$$

Similar to the uniaxial medium in our case the directions of the beam vector determining the flow of energy and the wave vector do not coincide. The angle δ between these vectors can be calculated if we take into account that the beam vector is normal to the surface of wave vectors. Thus, we have

$$\cos\,\delta = \frac{q_{st}\sin\theta + q_z\cos\theta}{\sqrt{q_{st}^2+q_z^2}}, \tag{34}$$

where θ is the angle between the z axis and vector $\mathbf{r}-\mathbf{r}_1$.

Figure 2 shows a cross section of a surface of beam vectors corresponding to the surface of a constant phase for a point source. It is seen that this surface has a break at $z=z_1$. This break corresponds to the forbidden zones. The normal to the beam surface determines the direction of a wave vector. The existence of forbidden zones limits the allowable angles between the normal to the beam surface and z axis by an angle γ_{max}. As a result we get the picture presented in Fig. 2.

IV. ORIGIN OF THE FORBIDDEN ZONE IN MEDIA WITH ONE-DIMENSIONAL LARGE-SCALE PERIODICITY

The existence of forbidden zones in media with large-scale inhomogeneities seems strange. Usually forbidden zones appear when the size of the inhomogeneities is of the order of the wavelength [16]. In order to understand the origin of the forbidden zones in our problem we introduce the refractive index

$$n(z) = \sqrt{1+\eta\,\cos(2\alpha z)}. \tag{35}$$

This value varies in the limits $\sqrt{1-\eta} \leqslant n \leqslant \sqrt{1+\eta}$. In the geometric optics approximation we introduce the concept of a beam trajectory described by the curve $\boldsymbol{\rho}=\boldsymbol{\rho}(z)$. Since the beam curve in a stratified isotropic medium is in the plane of incidence, we introduce for convenience the coordinate system with the y axis lying in the same plane. In each point $(\boldsymbol{\rho},z)$ the beam obeys the relation

$$n(z)\sin\,\chi(z) = C, \tag{36}$$

where C is the positive constant and $\chi(z)$ is the angle of incidence being counted from the plane normal to z, $0 \leqslant \chi \leqslant \pi/2$. Equality (36) is the Snells law. The angle $\chi(z)$ is related to the beam trajectory by the equation

$$\tan^2\chi(z) = [y'(z)]^2. \tag{37}$$

From Eqs. (36) and (37) we get

$$y'(z) = \pm\frac{C}{\sqrt{n^2(z)-C^2}}. \tag{38}$$

Here signs plus and minus correspond to parts of the beam where $y(z)$ increases or decreases with z, respectively. To be definite we choose the part with sign plus. As far as our problem is concerned $n(z)$ is the periodic function of z it follows from Eq. (38) that the derivative $y'(z)$ is a periodic function of z also. Performing a transition similar to that from Eq. (19) to Eq. (21) we can write

$$y(z) = g(z-z_0) + M(z), \tag{39}$$

where $M(z)$ is a periodic function with a zero average, z_0 is an arbitrary constant, and g is an average tangent of the slope of the $y(z)$ curve,

$$g = \frac{\alpha C}{\pi}\int_0^{\pi/\alpha}\frac{dz}{\sqrt{1+\eta\,\cos(2\alpha z)-C^2}} = \frac{2C}{\pi\sqrt{1+\eta-C^2}}\,K\left(\sqrt{\frac{2\eta}{1+\eta-C^2}}\right). \tag{40}$$

As long as the function $K(a)$ is real for $|a|<1$ the value of the C parameter is limited by the condition $C<C_* = \sqrt{1-\eta}$. According to Eq. (40) variation of C in the region $0 \leqslant C < \sqrt{1-\eta}$ leads to variation of g in the limits from 0 up to ∞ since the function $K(a)$ has a logarithmic divergence at $a\rightarrow 1$. Thus, changing the initial slope of the beam $\chi(z_1)$ in the point z_1 and hence the parameter C, we can observe beams with an arbitrary relation $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|/|z-z_1|$ at large $|z-z_1|$. This means that the surface of beam vectors, Fig. 2, has no forbidden zones. Figure 3 shows the trajectories of the beams outside of the forbidden zone.

In order to explain the emergence of a forbidden zone on the surface of wave vectors, Fig. 1, we introduce the wave

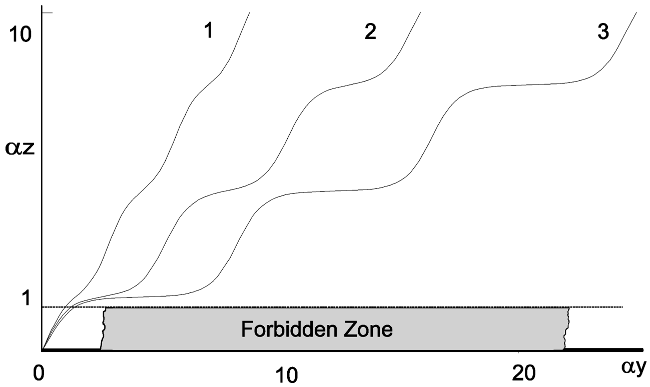

FIG. 3. Beam trajectories outside of the forbidden zone, $C < C_*$. The curves are obtained by numerical calculation for $\eta = 0.5$ and different values of C: 1, $C=0.6$; 2, $C=0.7$; 3, $C=0.707$.

vector $\mathbf{k}$ corresponding to the given beam. For this purpose we consider variation of the field phase along the beam between the points $\mathbf{r}_1=(y_1,z_1)$ and $\mathbf{r}=(y,z)$,

$$\Psi = k_0 \int_{\mathbf{r}_1}^{\mathbf{r}} n\,ds, \tag{41}$$

where s is the distance along the beam, $z>z_1$. Using the expression for an element of an arc $ds=\sqrt{1+[y'(z)]^2}dz$ and Eq. (38) we present integral (41) in the form

$$\Psi = k_0 C(y-y_1) + k_0 \int_{z_1}^{z} \sqrt{n^2(z)-C^2}\,dz. \tag{42}$$

According to Eq. (42) a transverse part of the wave vector $\mathbf{k}$ is equal to $\mathbf{k}_\perp=(0,k_y)$, $k_y=k_0C$. The longitudinal part of the wave vector k_z can be calculated if we take into account, that the phase (42) coincides with the phase of the wave in Eq. (25) at $C=q_{st}/k_0$. Using arguments similar to those in transition from Eq. (25) to Eq. (26) we get

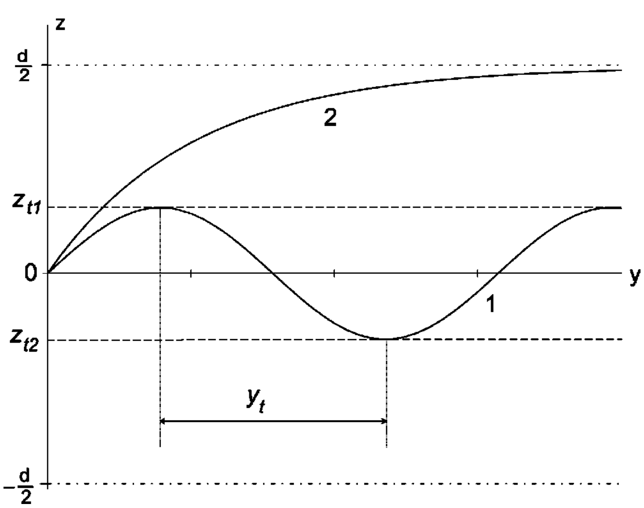

FIG. 4. Beam trajectories in the wave channel. Curve 1 corresponds to a trajectory at $(1-\eta)^{1/2}<C<(1+\eta)^{1/2}$; curve 2 is the limiting beam for $C=C_*=(1-\eta)^{1/2}$.

$$k_z = \frac{2k_0}{\pi}\sqrt{1+\eta-C^2}\,E\left(\sqrt{\frac{2\eta}{1+\eta-C^2}}\right). \tag{43}$$

It follows from Eq. (43) that limiting value $C=C_* = \sqrt{1-\eta}$ corresponds to the relation

$$\frac{k_\perp}{k_z} = \frac{\pi}{2}\sqrt{\frac{1-\eta}{2\eta}}, \tag{44}$$

which exactly coincides with the restriction imposed on the angle γ between $\mathbf{k}_{st}$ and the z axis in Eq. (33).

Thus, for the introduced wave vector $\mathbf{k}=(\mathbf{k}_\perp,k_z)$ there exists a forbidden zone since at $C>C_*$ the component k_z in Eq. (43) becomes complex and the wave starts to decay. The latter means that for such C the wave could not propagate at long distances z since the extinction is of the order of the wavelength.

V. FORBIDDEN ZONE AND 2D WAVEGUIDE

From the point of view of the geometric optics this effect can be explained by the turn (''reflection'') of the beam, which is possible if the refraction index decreases in the direction of the beam propagation [20]. As a result the beams with $C>C_*$ appear to be ''trapped'' within the limits of one layer (Fig. 4). The length of the beam trajectory between two successive reflections can be determined by integrating the right-hand side of Eq. (38). In order to describe wave propagation in the framework of geometric optics it is necessary to sew segments of a beam with indices plus and minus in Eq. (38) taking into account the $\pi/2$ phase change at each reflection. This approach describes the beam everywhere excluding the turn points [20].

The planes $z=z_t$ where the beams change their directions (are reflected) are determined by the condition $\sin\chi(z_t)=1$ in Eq. (36), i.e.,

$$n(z_1)\sin\chi(z_1)=n(z_t), \tag{45}$$

where z_1 is the position of the source, $|z_1|<d/2$. Equation (45) has two solutions,

$$z_{t1,2} = \pm\frac{1}{2\alpha}\arccos\frac{n^2(z_1)\sin^2\chi(z_1)-1}{\eta}. \tag{46}$$

The beam will propagate between planes $z=z_{t1}$ and $z=z_{t2}$, alternately being reflected from each of them.

So a plane channel of wave propagation is formed. The boundaries of this channel are the planes $z=d/2$ and $z=-d/2$ where the refractive index $n(z)$ is minimal. The waves trapped in this channel satisfy the condition

$$n(z_1)\sin\chi(z_1) \geqslant \sqrt{1-\eta} = C_*. \tag{47}$$

The width of the channel $z_t=|z_{t1}-z_{t2}|$ for a particular beam depends on the parameter C. The width $z_t\to 0$ for $C\to\sqrt{1+\eta}$ and $z_t\to d$ for $C\to\sqrt{1-\eta}$. Note, that in the limit $C\to\sqrt{1-\eta}$ sliding incidence takes place, and the beam approaches asymptotically the plane $z=d/2$ (or $z=-d/2$) without reflection. This beam is also locked in the wave channel (beam 2, Fig. 4).

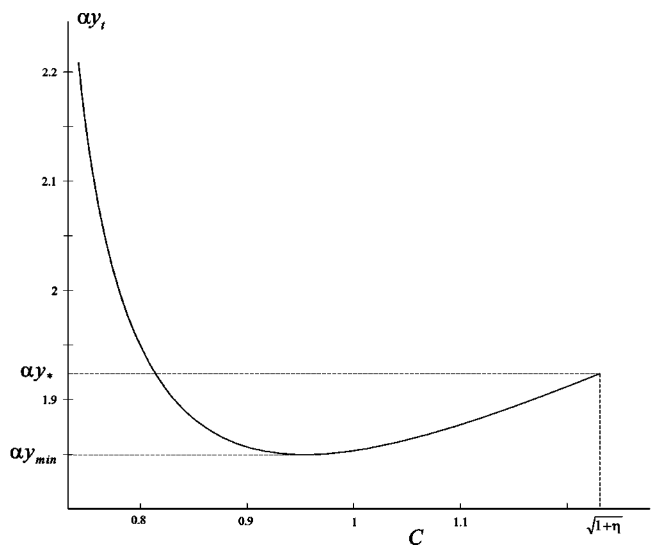

FIG. 5. Dependence of the distance passed by a beam between two reflections along the y axis calculated at $\eta=0.5$.

From Eq. (38) it is possible to get a distance passed by the beam between two successive reflections

$$y_t(z_1,\chi(z_1))$$

$$=n(z_1)\sin\,\chi(z_1)\int_{z_{t2}}^{z_{t1}}\frac{dz}{\sqrt{n^2(z)-n^2(z_1)\sin^2\chi(z_1)}}. \quad (48)$$

Expressions (46) and (48) show that z_t and y_t may be considered as functions of $C=n(z_1)\sin\,\chi(z_1)$. At $C\to\sqrt{1-\eta}$ the distance $y_t\to\infty$. With increasing of C the function y_t decreases, passing through its minimum, y_{min}, and $y_*=d\sqrt{(1+\eta)/2\eta}$ at $z_t\to 0$, i.e., for $C=\sqrt{1+\eta}$ (Fig. 5).

First, we estimate the asymptotics of the Green's function at $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$ based on simple physical grounds. The waves, satisfying the condition (47), will remain in the plane layer $-d/2\leq z\leq d/2$, at any $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|$. Therefore the energy density of these waves will decrease as $|\mathbf{r}-\mathbf{r}_1|^{-1}\approx|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^{-1}$ with increasing the distance from the source. Hence the amplitude of the field of a point source behaves as $|\mathbf{r}-\mathbf{r}_1|^{-1/2}$ at $|\mathbf{r}-\mathbf{r}_1|\to\infty$ when points z and z_1 are in the same wave channel. It differs from the usual situation $|z-z_1|\to\infty$, when, according to Eq. (25), the amplitude of a field decreases as $|\mathbf{r}-\mathbf{r}_1|^{-1}$.

In order to find a field of a point source in the wave channel it is necessary to take into account all orders of reflection [20]. The field inside the wave channel may be written as

$$T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\sum_N T_N(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1), \quad (49)$$

where $T_N(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)$ is the contribution to the Green's function of waves undergoing N reflections. This contribution differs from Eq. (16) as long as in this case it is necessary to take into account the contribution of multiple reflections to the resulting phase:

$$T_N(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\sum_{m=1}^{2}\int d\mathbf{q}\,\frac{i\,\exp\left[i\left(\Omega N\int_{\alpha z_{t2}(q)}^{\alpha z_{t1}(q)}d\xi\Gamma(q,\xi)+\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)-\frac{N\pi}{2}+\delta\Psi_N^{(m)}\right)\right]}{8\pi^2k_0\Gamma^{1/2}(q,\alpha z)\Gamma^{1/2}(q,\alpha z_1)}. \quad (50)$$

Values $\delta\Psi_N^{(m)}=\delta\Psi_N^{(m)}(z,z_1,q)$ are contributions to the total phase of distances between z_1 and the first reflection and between the last reflection and z, respectively. If the first reflection occurs in the plane $z=z_{t1}$ the index $m=1$, while if it occurs in the plane $z=z_{t2}$ index $m=2$. Functions $z_{t1}(q)$ and $z_{t2}(q)$ are determined by Eq. (46) with $C=q/k_0$. Number N in Eq. (49) varies in the limits $N=0$ and $N=|\boldsymbol{\rho}-\boldsymbol{\rho}_1|/y_{min}$.

At $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$ integral (50) can be calculated by the method of stationary phase, as it was in the case of the integral in Eq. (16). The equation determining the stationary point q_{stN} has the form

$$|\boldsymbol{\rho}-\boldsymbol{\rho}_1|=Ny_t(q_{stN})+\delta\rho_N^{(m)}, \quad (51)$$

where $\delta\rho_N^{(m)}=\delta\rho_N^{(m)}(z,z_1,q_{stN})$ is a sum of distances passed by the wave along the ρ axis (i) from the source $\mathbf{r}_1$ to the first reflection and (ii) from the point of the last reflection to the observation point $\mathbf{r}$. The function $y_t(q)$ is determined by Eq. (48) with $C=q/k_0$. Then Eq. (50) can be written as

$$T_N(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\sum_{q_{stN}}T(q_{stN};\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1), \quad (52)$$

where

$$T(q;\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=-\sigma q^{1/2}y_t^{1/2}(q)|y_t'(q)|^{-1/2}\,\frac{\exp\left[i|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\left(q+k_0y_t^{-1}(q)\left(\int_{\alpha z_{t2}(q)}^{\alpha z_{t1}(q)}d\xi\Gamma(q,\xi)-\frac{\lambda}{4}\right)\right)\right]\sum_{m=1}^{2}\exp(i\delta\Psi_N^{(m)})}{4\pi k_0|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\Gamma^{1/2}(q,\alpha z)\Gamma^{1/2}(q,\alpha z_1)}. \quad (53)$$

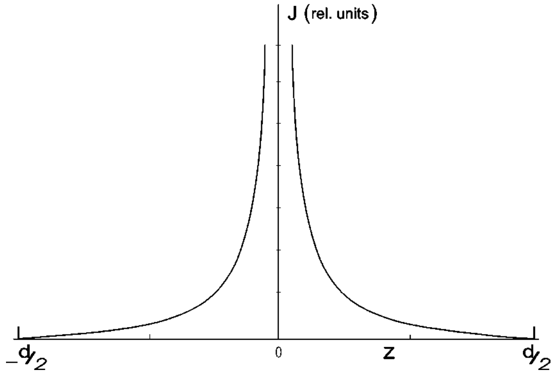

FIG. 6. Distribution of intensity $J=\langle|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2\rangle|\boldsymbol{\rho}-\boldsymbol{\rho}_1|$ inside the wave channel at $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$, for the source at the point $(\rho,0)$, calculated at $\eta=0.5$.

The summation over q_{stN} in Eq. (52) takes into account the possibility of the presence of several waves for any set (N,m). It is seen from Fig. 5 that period y_t can correspond to two different values of the parameter C. According to Eq. (51) this means that one set of (N,m) can correspond to two different values of q_{stN}, i.e., two waves radiated from the source.

We study the z dependence of intensity proportional to $|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2$ at $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$ in the wave channel $-d/2\leqslant z\leqslant d/2$. For further analysis it is convenient to pass from the summation over N and q_{stN} to the summation over stationary points $q_{st}^{(i)}$, corresponding to all waves, radiated in $\mathbf{r}_1$ and transmitted to the observation point $\mathbf{r}$. The set of $q_{st}^{(i)}$, $i=1,2,\ldots$ is enumerated in increasing order of values. We have

$$|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2=\sum_{q_{st}^{(i)}} T(q_{st}^{(i)};\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)\sum_{q_{st}^{(j)}} T^*(q_{st}^{(j)};\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1). \tag{54}$$

The experimental measurements of intensity always imply a procedure of averaging over time, sizes of the source and the receiver, random inhomogeneities, etc. Therefore the contributions of terms with a large difference $\Delta q=q_{st}^{(i)}-q_{st}^{(j)}$ may be omitted since their phases are not correlated. Preserving in Eq. (54) only terms with small difference Δq and taking into account that the averaged module of $T(q;\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)$ is a smooth function of q, the average intensity $\langle|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2\rangle$ may be written as

$$\langle|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2\rangle=\sum_{q_{st}^{(i)}} |T(q_{st}^{(i)};\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2 D(q_{st}^{(i)},\boldsymbol{\rho}-\boldsymbol{\rho}_1,z,z_1), \tag{55}$$

where

$$D(q_{st}^{(i)},\boldsymbol{\rho}-\boldsymbol{\rho}_1,z,z_1)=\frac{1}{2}\sum_{m=1}^{2}\sum_{\Delta q_{(l)}}\langle\exp[i\Delta\Psi_m^{(l)}(q_{st}^{(i)})]\rangle \tag{56}$$

is the factor taking into account the interference effects between waves with close numbers of reflections. Here $\Delta q_{(l)}=\Delta q_{(l)}(q_{st}^{(i)})=q_{st}^{(l+i)}-q_{st}^{(i)}$ and $\Delta\Psi_m^{(l)}(q_{st}^{(i)})=\Psi_m(q_{st}^{(l+i)})-\Psi_m(q_{st}^{(i)})$ (the latter being the phase differences of waves with wave numbers $q_{st}^{(l+i)}$ and $q_{st}^{(i)}$). As long as only the terms with small l interfere, it is possible for the summation over $\Delta q_{(l)}$ in the function D to spread from $-\infty$ up to $+\infty$.

Since the terms with $m=1,2$ do not interfere we get from Eq. (53)

$$|T(q;\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2=\frac{q y_t(q)}{8\pi^2k_0^2\Gamma(q,\alpha z)\Gamma(q,\alpha z_1)|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2|y_t{}'(q)|}. \tag{57}$$

Let us consider a wave with a wave number $q_{st}^{(i)}$. The number of reflections undergone by the wave N is a function of $q_{st}^{(i)}$: $N=N(q_{st}^{(i)})$. For a wave with the nearest wave number $q_{st}^{(i-1)}$ the number of reflections will change by 1. Neglecting small additions $\delta\rho_N^{(m)}$ we have from Eq. (51)

$$\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{y_t(q_{st}^{(i)})}-\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}{y_t(q_{st}^{(i-1)})}=1. \tag{58}$$

Hence for $\delta q=q_{st}^{(i-1)}-q_{st}^{(i)}$ we get

$$\delta q=\frac{y_t^2(q_{st}^{(i)})}{y_t'(q_{st}^{(i)})|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}. \tag{59}$$

It is seen, that with the increase of $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|$ the size $\delta q\to 0$. Therefore in the limit $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$ it is possible in Eq. (55) to pass from summation over $q_{st}^{(i)}$ to integration over q. Then Eq. (55) is written as

$$\langle|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2\rangle=\int_{k_0\sqrt{1-\eta}}^{k_0\min[n(z_1);n(z)]}\frac{D(q,z,z_1)q\,dq}{8\pi^2k_0^2\Gamma(q,\alpha z)\Gamma(q,\alpha z_1)|\boldsymbol{\rho}-\boldsymbol{\rho}_1|y_t(q)}, \tag{60}$$

or, passing to a variable $C=q/k_0$, we get

$$\langle|T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)|^2\rangle=\frac{1}{16\pi^2|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}\int_{\sqrt{1-\eta}}^{\min[n(z_1);n(z)]}\frac{D(k_0C,z,z_1)dC}{\sqrt{n^2(z_1)-C^2}\sqrt{n^2(z)-C^2}}\left(\int_0^{n(z')=C}\frac{dz'}{\sqrt{n^2(z')-C^2}}\right)^{-1}. \tag{61}$$

It is seen from Eq. (61), that $T(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)\sim|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^{-(1/2)}$ at $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\to\infty$ according to physical reasons discussed earlier.

Figure 6 shows the distribution of the radiated energy locked in the wave channel along the z axis. For this purpose integral (61) is calculated at $D=1$, which corresponds to neglect of interference between waves with different numbers of reflections. On boundaries of the wave channel the intensity tends to zero, as the area of integration in Eq. (61) vanishes. Here we should attract attention to the presence of a sharp peak at $z=z_1$. Formally it appears due to confluence in Eq. (61) of root type peculiarities at $n(z)=n(z_1)$. In this case the integrand has a pole. It means that the intensity decreases more slowly than $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^{-1}$. The physical reason for such a change of asymptotic at $n(z)=n(z_1)$ is that both the reception and radiation points are situated in the caustic regions. To remove the divergence and to obtain the asymptotic a more detailed analysis of the field in the vicinity of the caustic is required.

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